

# K-THEORY OF EQUIVARIANT QUANTIZATION

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ABSTRACT. We prove that equivariant  $K$ -theory is invariant under strict deformation quantization for a compact Lie group action.

## 1. INTRODUCTION

Let  $\alpha$  be a strongly continuous action of  $\mathbb{R}^n$  on a separable  $C^*$ -algebra  $A$ , and  $J$  be a skew-symmetric matrix on  $\mathbb{R}^n$ . Rieffel [8] constructed a strict deformation quantization  $A_J$  of  $A$  via oscillatory integrals

$$(1) \quad a \times_J b := \int_{\mathbb{R}^n \times \mathbb{R}^n} \alpha_{Ju}(a) \alpha_v(b) e^{2\pi i u \cdot v} du dv,$$

for  $u, v \in \mathbb{R}^n$ , and  $a, b \in A^\infty$  (the smooth subalgebra of  $A$  for  $\alpha$ ). Such a construction gives rise to many interesting examples of noncommutative manifolds, e.g. quantum tori,  $\theta$ -deformation of  $S^4$ , etc. In [9], Rieffel proved that the  $K$ -theory of  $A_J$  is equal to the  $K$ -theory of the original algebra  $A$ .

In this paper, we are interested in examples that the algebra  $A$  is also equipped with a strongly continuous action  $\beta$  by a compact group  $G$ . When the two actions commute, Rieffel's results naturally generalize to the equivariant setting. An easy observation is that, as the  $G$ -action  $\beta$  commutes with the  $\mathbb{R}^n$ -action  $\alpha$ , naturally  $\alpha$  can be lifted to a strongly continuous action  $\tilde{\alpha}$  on the crossed product algebra  $A \rtimes_\beta G$ . Rieffel's construction (1) applies to the  $\mathbb{R}^n$ -action  $\tilde{\alpha}$  on  $A \rtimes_\beta G$ , and defines a quantization algebra  $(A \rtimes_\beta G)_J$ . By the commutativity between  $\alpha$  and  $\beta$ , we easily check that  $\beta$  lifts to a strongly continuous action  $\tilde{\beta}$  on  $A_J$ , and  $A_J \rtimes_{\tilde{\beta}} G$  is isomorphic to  $(A \rtimes_\beta G)_J$ . Now by Rieffel's theorem on the  $K$ -theory of strict deformation quantization [9], we conclude that

$$K_\bullet(A \rtimes_\beta G) = K_\bullet((A \rtimes_\beta G)_J) = K_\bullet(A_J \rtimes_{\tilde{\beta}} G).$$

In this paper, we generalize the above discussion of equivariant quantization to the situation where the actions  $\alpha$  and  $\beta$  do not commute. Define  $GL(J)$  to be the group of invertible matrices  $g$  such that  $g^t J g = J$ , and  $SL_n(\mathbb{R}, J) := SL_n(\mathbb{R}) \cap GL(J)$ . We remark that when  $J$  is the standard skew-symmetric matrix on  $\mathbb{R}^{2n}$ ,  $GL(J)$  is the linear symplectic group. Let  $\rho : G \rightarrow SL_n(\mathbb{R}, J)$  be a group homomorphism, and

$$(2) \quad \beta_g \alpha_x = \alpha_{\rho_g(x)} \beta_g, \quad \text{for any } g \in G, x \in \mathbb{R}^n.$$

When  $\rho$  is a trivial group homomorphism, the actions  $\alpha$  and  $\beta$  commute.

A natural example of such a system appears as follows.

**Example 1.1.** Let  $G = \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$  act on  $\mathbb{R}^{2n}$  by reflection. Let  $\mathbb{Z}^{2n}$  be the integer lattice in  $\mathbb{R}^{2n}$ . The  $2n$ -torus  $\mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$  inherits an action of  $\mathbb{Z}_2$  from the  $\mathbb{Z}_2$  action on  $\mathbb{R}^{2n}$ . The group  $\mathbb{R}^{2n}$  acts on  $\mathbb{R}^{2n}$  by translation and descends to act on  $\mathbb{T}^{2n}$ . Let  $A$  be the  $C^*$ -algebra of continuous functions on  $\mathbb{T}^{2n}$ , and  $J$  be the standard symplectic matrix on  $\mathbb{R}^{2n}$ . The action  $\alpha$  (and  $\beta$ ) of  $\mathbb{R}^{2n}$  (and  $\mathbb{Z}_2$ ) on  $A$  is the dual action of the corresponding actions on  $\mathbb{T}^{2n}$ . We easily check that Eq. (2) holds in this case with  $\rho$  being the natural inclusion  $\mathbb{Z}_2 \hookrightarrow SL_{2n}(\mathbb{R}, J)$ .

Different from the case where the actions  $\alpha$  and  $\beta$  commute, for a nontrivial  $\rho : G \rightarrow SL_n(\mathbb{R}, J)$ , the  $\mathbb{R}^n$ -action  $\alpha$  on  $A$  does not lift naturally to an action on  $A \rtimes_\beta G$ . Therefore, we cannot apply Rieffel's deformation construction to the algebra  $A \rtimes_\beta G$ . Nevertheless, a simple calculation shows that

$$\beta_g(a \times_J b) = \beta_g(a) \times_J \beta_g(b), \quad \beta_g(a^*) = \beta_g(a)^*,$$

which shows that the  $G$ -action  $\beta$  is still well-defined on  $A_J$ . Accordingly, we can consider the crossed product algebra  $A_J \rtimes_\beta G$ . Applying this construction to Ex. 1.1, we obtain  $A_J \rtimes_\beta \mathbb{Z}_2$ , which is well studied in literature, e.g. [4] and [5].

In this paper, we prove the following theorem about the  $K$ -theory groups of  $A_J \rtimes_\beta G$ .

**Theorem 1.2.** *If the actions  $\alpha$ ,  $\beta$  and the group homomorphism  $\rho$  satisfy (2), then*

$$K_\bullet(A_J \rtimes_\beta G) \cong K_\bullet(A \rtimes_\beta G), \quad \bullet = 0, 1.$$

The proof of this theorem will be presented in the next section. As applications of our theorem, we recover some results of [4] on the computation of the  $K$ -groups of  $\mathbb{Z}_i$ -quantum tori for  $i = 2, 3, 4, 6$ , and we generalize these results to the  $\theta$ -deformation [3] of  $S^4$ .

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## 2. PROOF OF THE MAIN THEOREM

Our proof of Theorem 1.2 is an equivariant generalization of Rieffel's original proof in [9]. Following Rieffel's method [9], we will decompose our proof into 3 steps.

**Step I.** Following Rieffel's notation [9], we let  $\mathcal{B}^A$  be the space of smooth  $A$ -valued functions on  $\mathbb{R}^n$  whose derivatives together with themselves are bounded on  $\mathbb{R}^n$ . Let  $\mathcal{S}^A$  be the space of  $A$ -valued Schwartz functions on  $\mathbb{R}^n$ . The integral

$$\langle f, g \rangle_A := \int f(x)^* g(x) dx$$

defines an  $A$ -valued inner product on  $\mathcal{S}^A$ . Given  $J$ , we define a product on  $\mathcal{B}^A$

$$(F \times_J G)(x) := \int F(x + Ju) G(x + v) e^{2\pi i u \cdot v} du dv, \quad F, G \in \mathcal{B}^A.$$

Furthermore,  $\mathcal{B}^A$  acts on  $\mathcal{S}^A$  by

$$(L_F^J f)(x) := \int F(x + Ju) f(x + v) e^{2\pi i u \cdot v} du dv, \quad F \in \mathcal{B}^A, f \in \mathcal{S}^A.$$

Via the  $A$ -valued inner product on  $\mathcal{S}^A$ , we can equip  $\mathcal{B}^A$  with the operator norm  $\| \cdot \|_J$ , and obtain a pre- $C^*$ -algebra  $(\mathcal{B}_J^A, \times_J, \| \cdot \|_J)$ . Denote the corresponding  $C^*$ -algebra by  $\overline{\mathcal{B}}_J^A$ . Meanwhile,  $\mathcal{S}^A$  viewed as a  $*$ -ideal of  $\mathcal{B}_J^A$  (cf. Rieffel, [8]), denoted by  $\mathcal{S}_J^A$ , can be completed into  $\overline{\mathcal{S}}_J^A$ .

With an action  $\alpha$  of  $\mathbb{R}^n$  on  $A$ , Rieffel [9, Prop. 1.1] introduced a strongly continuous  $\mathbb{R}^n$ -action  $\nu$  on  $\overline{\mathcal{B}}_J^A$  and also on  $\overline{\mathcal{S}}_J^A$  by

$$(\nu_t(F))(x) := \alpha_t(F(x - t)).$$

The fixed point subalgebra of this action  $\nu$  is identified [9, Prop. 2.14] with the  $C^*$ -subalgebra of  $\overline{\mathcal{S}}_J^A$  generated by elements

$$\tilde{a}(x) := \alpha_x(a), \quad a \in A^\infty,$$

which is exactly  $A_J$ .

Rieffel [9, Thm. 3.2] proved that  $A_J$  is strongly Morita equivalent to  $\overline{\mathcal{S}}_J^A \rtimes_\nu \mathbb{R}^n$ . We will generalize this theorem to the equivariant setting with the  $G$ -action  $\beta$ . We introduce the  $G$ -action  $\overline{\beta}$  on  $\mathcal{B}_J^A$  by

$$\overline{\beta}_g(F)(x) := \beta_g(F(g^{-1}(x))).$$

The exactly same arguments as [9, Prop. 1.1] prove that the  $G$ -action  $\overline{\beta}$  is strongly continuous on  $\mathcal{S}^A$ , therefore so is it on  $\overline{\mathcal{S}}_J^A$ .

**Proposition 2.1.** *The crossed product algebras  $A_J \rtimes_\beta G$  and  $(\overline{\mathcal{S}}_J^A \rtimes_\nu \mathbb{R}^n) \rtimes_{\overline{\beta}} G$  are strongly Morita equivalent.*

*Proof.* We will apply Combes' theorem [1, Sec. 6] on equivariant Morita equivalence. We will prove that the  $G$ -actions  $\beta$  and  $\overline{\beta}$  are Morita equivalent, which by Combes' theorem implies the Morita equivalence we seek.

According to [1], two  $G$ -actions  $\beta^1, \beta^2$  on  $A$  and  $B$  are Morita equivalent if there is a strong Morita equivalence bimodule  $X$  between  $A$  and  $B$  such that there is a  $G$ -action  $\beta$  on  $X$  satisfying

$$\begin{aligned} \beta_g(a\xi) &= \beta_g^1(a)\beta_g(\xi), & \beta_g(\xi b) &= \beta_g(\xi)\beta_g^2(b), \\ A\langle\beta_g(\xi_1), \beta_g(\xi_2)\rangle &= \beta_g^1(A\langle\xi_1, \xi_2\rangle), & \langle\beta_g(\xi_1), \beta_g(\xi_2)\rangle_B &= \beta_g^2(\langle\xi_1, \xi_2\rangle_B). \end{aligned}$$

for  $\xi, \xi_1, \xi_2 \in X$ .

Rieffel [9] constructed a Morita equivalence bimodule between  $A_J$  and  $\overline{\mathcal{S}}_J^A \rtimes_\nu \mathbb{R}^n$ . We recall it now. Let  $C_\infty(\mathbb{R}^n, A)$  be the  $C^*$ -algebra of  $A$ -valued functions on  $\mathbb{R}^n$  that vanish at infinity. Let  $\tau$  be the  $\mathbb{R}^n$ -action on  $\mathcal{B}_J^A$  by translation,  $(\tau_t F)(x) = F(t + x)$ , and  $\mu$  be the action of  $\mathbb{R}^n$  on  $C_\infty(\mathbb{R}^n, A)$  by

$$\mu_s(f)(x) = e^{2\pi i s \cdot x} f(x).$$

Define an action  $\alpha$  of  $\mathbb{R}^n$  on  $\overline{\mathcal{S}}_J^A$  by

$$\alpha_t(F)(x) = \alpha_t(F(x)).$$

Both  $\mu$  and  $\tau$  act on  $C_\infty(\mathbb{R}^n, A)$  and their combination gives an action of the Heisenberg group  $H$  of dimension  $2n + 1$  on  $C_\infty(\mathbb{R}^n, A)$ . This Heisenberg group action commutes with  $\alpha$  and defines an  $H \times \mathbb{R}^n$ -action  $\sigma$  on  $C_\infty(\mathbb{R}^n, A)$ . Define  $X_0$  to be the subspace of  $C_\infty(\mathbb{R}^n, A)$  of  $\sigma$ -smooth vectors, and a suitable completion  $\overline{X}_0$  of  $X_0$  serves as a strong Morita equivalence bimodule, which we refer to [9] for details.

The algebra  $\overline{\mathcal{S}}_J^A$  acts on  $\overline{X}_0$  by right multiplication. Define a right  $A_J$ -module structure on  $\overline{X}_0$  by identifying  $A_J$  with the subspace of  $\nu$ -invariant vectors in  $\overline{\mathcal{S}}_J^A$ . The algebra  $\overline{\mathcal{S}}_J^A \rtimes_\nu \mathbb{R}^n$  acts on  $\overline{X}_0$  by

$$\psi(f) := \int \psi(t) \times_J \nu_t(f) dt, \quad \psi \in \overline{\mathcal{S}}_J^A \rtimes_\nu \mathbb{R}^n, \quad f \in \overline{X}_0.$$

We define an  $\overline{\mathcal{S}}_J^A \rtimes_\nu \mathbb{R}^n$ -valued inner product on  $\overline{X}_0$  by

$$\overline{\mathcal{S}}_J^A \rtimes_\nu \mathbb{R}^n \langle f, g \rangle(x) := f \times_J \nu_x(g^*), \quad x \in \mathbb{R}^n, \quad f, g \in \overline{X}_0,$$

and an  $A_J$ -valued inner product on  $\overline{X}_0$  by

$$\langle f, g \rangle_{A_J}(x) := \left( \int \nu_t(f^* \times_J g) dt \right) (x) = \left( f^* \times_J \nu_x \left( \int \nu_t g(-t) dt \right) \right) (x), \quad f, g \in \overline{X}_0.$$

Rieffel [9] proved that  $(\overline{X}_0, \overline{\mathcal{S}}_J^A \rtimes_{\nu} \mathbb{R}^n \langle \cdot, \cdot \rangle, \langle \cdot, \cdot \rangle_{A_J})$  is a strong Morita equivalence bimodule between  $\overline{\mathcal{S}}_J^A \rtimes_{\nu} \mathbb{R}^n$  and  $A_J$ .

We easily check the following identities between the actions

$$\overline{\beta}_g \alpha_t = \alpha_{g(t)} \overline{\beta}_g, \quad \overline{\beta}_g \tau_t = \tau_{g(t)} \overline{\beta}_g, \quad \overline{\beta}_g \mu_t = \mu_{(\rho(g)^T)^{-1}(t)} \overline{\beta}_g, \quad g \in G, \quad t \in \mathbb{R}^n.$$

where  $\rho(g)^T$  is the transpose of  $\rho(g)$ . These identities show that the  $G$ -action  $\overline{\beta}$  on  $C_{\infty}(\mathbb{R}^n, A)$  preserves the subspace  $X_0$  of  $\sigma$ -smooth vectors. Using the property that  $\beta$  and  $\overline{\beta}$  act strongly continuously on  $\overline{\mathcal{S}}_J^A$ , we can easily check that  $\beta$  and  $\overline{\beta}$  are Morita equivalent  $G$ -actions in the sense of Combes [1]. Therefore,  $A_J \rtimes_{\beta} G$  is strongly Morita equivalent to  $(\overline{\mathcal{S}}_J^A \rtimes_{\nu} \mathbb{R}^n) \rtimes_{\overline{\beta}} G$ .  $\square$

By the Morita equivalence, we conclude that

$$K_{\bullet}(A_J \rtimes_{\beta} G) \cong K_{\bullet}((\overline{\mathcal{S}}_J^A \rtimes_{\nu} \mathbb{R}^n) \rtimes_{\overline{\beta}} G).$$

**Step II.** Let  $\mathbb{C}_n$  be the complex Clifford algebra associated with  $\mathbb{C}^n$ . We use the following equivariant Thom isomorphism Theorem<sup>1</sup> due to Kasparov [6, Thm. 2.].

**Theorem 2.1.** *Let  $\mathbb{R}^n$  and  $G$  act strongly continuously on a separable  $C^*$ -algebra  $B$  with the actions denoted by  $\alpha$  and  $\beta$ . Let  $\rho : G \rightarrow GL(n, \mathbb{R})$ . If the actions  $\alpha$  and  $\beta$  satisfy Equation (2), then*

$$K_{\bullet}(((B \otimes \mathbb{C}_n) \rtimes_{\alpha} \mathbb{R}^n) \rtimes_{\beta} G) \cong K_{\bullet}^G((B \otimes \mathbb{C}_n) \rtimes_{\alpha} \mathbb{R}^n) \cong K_{\bullet}^G(B) \cong K_{\bullet}(B \rtimes_{\beta} G),$$

where  $\mathbb{C}_n$  is the complex Clifford algebra associated with  $\mathbb{C}^n$ .

Taking  $B = \overline{\mathcal{S}}_J^A \otimes \mathbb{C}_n$  in the above theorem, we conclude that  $K_{\bullet}((\overline{\mathcal{S}}_J^A \otimes \mathbb{C}_n) \rtimes_{\beta} G)$  is isomorphic to  $K_{\bullet}((\overline{\mathcal{S}}_J^A \rtimes_{\nu} \mathbb{R}^n) \rtimes_{\overline{\beta}} G)$ .

**Step III.** Rieffel proved [8, Prop. 5.2] that there is an isomorphism

$$(3) \quad \overline{\mathcal{S}}_J^A \cong A \otimes \mathcal{K} \otimes C_{\infty}(V_0),$$

where  $\mathcal{K}$  is the algebra of compact operators on an infinite dimensional separable Hilbert space  $\mathcal{H}$ , and  $V_0$  is the kernel of  $J$  in  $\mathbb{R}^n$ . Let  $U$  be the orthogonal complement of  $V_0$  in  $\mathbb{R}^n$ . It is easy to check that  $U$  is a  $J$ -invariant subspace, and both  $U$  and  $V_0$  are  $G$ -invariant subspaces. As  $G$  is compact, there is a  $G$ -invariant complex structure on  $U$  compatible with  $J|_U$  (viewed as a symplectic form on  $U$ ). Without loss of generality, we will just assume that  $G$  preserves the standard complex structure on  $U$ . The key observation in the proof of [8, Prop. 5.2] is that when  $A$  is the trivial  $C^*$ -algebra  $\mathbb{C}$  and  $J$  invertible,  $\overline{\mathcal{S}}_J^{\mathbb{C}}$  is naturally identified as the space of compact operators, still denoted by  $\mathcal{K}$ , on the subspace  $\mathcal{H}$  of  $L^2(U)$  generated by elements

$$g(\bar{z}) e^{-\frac{\|z\|^2}{2}},$$

where  $g$  is an anti-holomorphic function. As  $\mathcal{H}$  is a  $G$ -invariant subspace, we can conclude that Rieffel's isomorphism (3) is  $G$ -equivariant (note that  $G$  acts on  $\mathcal{K}$  by conjugation). By Combes' result on  $G$ -equivariant Morita equivalence,  $(A \otimes \mathcal{K} \otimes C_{\infty}(V_0) \otimes \mathbb{C}_n) \rtimes_{\beta} G$  is strongly Morita equivalent to  $(A \otimes C_{\infty}(V_0) \otimes \mathbb{C}_n) \rtimes_{\overline{\beta}} G$ .

<sup>1</sup>In [6, Thm. 2.], an extra assumption that the group  $G$  is connected is assumed. But this assumption can be easily dropped using the same idea of the proof.

Now we look at the decomposition of  $\mathbb{R}^n$  as  $V_0 \oplus U$ . The Clifford algebra  $\mathbb{C}_n$  associated with  $\mathbb{C}^n$  is  $G$ -equivariantly isomorphic to  $\mathbb{C}_{V_0} \otimes \mathbb{C}_U$ , where  $\mathbb{C}_{V_0}$  and  $\mathbb{C}_U$  are the complex Clifford algebras associated with  $V_0$  and  $U$ , respectively. Notice that  $J$  restricts to define a symplectic form on  $U$ , and that the action of  $G$  preserves both the restricted  $J$  and the metric on  $U$ . Therefore the  $G$ -action on  $U$  is  $spin^c$ . Hence, the algebra  $(A \otimes C_\infty(V_0) \otimes \mathbb{C}_n) \rtimes_{\bar{\beta}} G$  is  $KK$ -equivalent to  $(A \otimes C_\infty(V_0) \otimes \mathbb{C}_{V_0}) \rtimes_{\bar{\beta}} G$ . Again by the  $G$ -equivariant Thom isomorphism Thm. 2.1 for the trivial  $V_0$  action on  $A$ , we conclude that

$$\begin{aligned} K_\bullet((\bar{\mathcal{S}}_J^A \otimes \mathbb{C}_n) \rtimes_{\bar{\beta}} G) &= K_\bullet((A \otimes \mathcal{K} \otimes C_\infty(V_0) \otimes \mathbb{C}_n) \rtimes_{\bar{\beta}} G) \\ &= K_\bullet((A \otimes C_\infty(V_0) \otimes \mathbb{C}_{V_0}) \rtimes_{\bar{\beta}} G) = K_\bullet(A \rtimes_{\beta} G). \end{aligned}$$

Summarizing Step I-III, we have the following equality,

$$K_\bullet(A_J \rtimes_{\beta} G) \stackrel{\text{Step I}}{=} K_\bullet((\bar{\mathcal{S}}_J^A \rtimes_{\nu} \mathbb{R}^n) \rtimes_{\bar{\beta}} G) \stackrel{\text{Step II}}{=} K_\bullet((\bar{\mathcal{S}}_J^A \otimes \mathbb{C}_n) \rtimes_{\bar{\beta}} G) \stackrel{\text{Step III}}{=} K_\bullet(A \rtimes_{\beta} G).$$

### 3. EXAMPLES

In this section, we discuss some applications of Theorem 1.2.

**3.1. Noncommutative toroidal orbifolds.** We identify a 2-torus  $\mathbb{T}^2$  by  $\mathbb{R}^2/\mathbb{Z}^2$ .  $\mathbb{R}^2$  acts on itself by translation and induces an action  $\alpha$  on  $\mathbb{T}^2$ . For  $\theta \in \mathbb{R}$ , we consider the symplectic form  $J = \theta dx_1 \wedge dx_2$  on  $\mathbb{R}^2$ . The group  $SL_2(\mathbb{Z})$  acts on  $\mathbb{R}^2$  preserving the lattice  $\mathbb{Z}^2$  and therefore also acts on  $\mathbb{T}^2$ , which is denoted by  $\beta$ . Inside  $SL_2(\mathbb{Z})$ , there are cyclic subgroups generated by

$$\begin{aligned} \sigma_2 &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, & \sigma_3 &= \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \\ \sigma_4 &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & \sigma_6 &= \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}. \end{aligned}$$

The element  $\sigma_i$  generates a cyclic subgroup  $\mathbb{Z}_i$  of  $SL_2(\mathbb{Z})$  of order  $i = 2, 3, 4, 6$ . In this example, the group  $SL_2(\mathbb{R}, J)$  is identical to the group  $SL_2(\mathbb{R})$ . Define  $\rho : \mathbb{Z}_i \rightarrow SL_2(\mathbb{R})$  to be the inclusion. And it is straightforward to check the actions  $\beta$  of  $\mathbb{Z}_i$  on  $\mathbb{T}^2$ ,  $\rho$  of  $\mathbb{Z}_i$  on  $\mathbb{R}^2$ , and  $\alpha$  of  $\mathbb{R}^2$  on  $\mathbb{T}^2$  satisfy Eq. (2). As is explained in Sec. 1, the group  $\mathbb{Z}_i$  naturally acts on Rieffel's deformation  $A_J$ , which is the quantum torus  $A_\theta$ . Theorem 1.2 states that

$$K_\bullet(A_J \rtimes \mathbb{Z}_i) = K_\bullet(A \rtimes \mathbb{Z}_i).$$

We recover with a completely different proof the result of [4, Cor. 2.2]. We have brought the question of computation of  $K$ -groups of these noncommutative orbifolds to a purely topological setting, and we refer to [4] and references therein for the explicit computation of the  $K$ -groups of the undeformed algebras  $A \rtimes \mathbb{Z}_i$ ,  $i = 2, 3, 4, 6$ . For example, when  $i = 2$ , the  $K$ -groups of  $A \rtimes \mathbb{Z}_2$  are

$$K_\bullet(A \rtimes \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}^6 & \bullet = 0, \\ 0 & \bullet = 1. \end{cases}$$

**3.2. Theta deformation.** Consider a 4-sphere  $S^4$  centered at  $(0, 0, 0, 0, 0)$  in  $\mathbb{R}^5$  with radius 1. In coordinates, it is the set

$$\{(x_1, \dots, x_5) | x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 1\}.$$

Defines  $\mathbb{T}^2$ -action on  $S^4$  by, for  $0 \leq t_1, t_2 < 2\pi$ ,

$$((t_1, t_2), (x_1, \dots, x_5)) \longrightarrow (x_1, \dots, x_5) \begin{pmatrix} \cos(t_1) & \sin(t_1) & 0 & 0 & 0 \\ -\sin(t_1) & \cos(t_1) & 0 & 0 & 0 \\ 0 & 0 & \cos(t_2) & \sin(t_2) & 0 \\ 0 & 0 & -\sin(t_2) & \cos(t_2) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The same formula as above also defines an  $\mathbb{R}^2$ -action  $\alpha$  on  $S^4$ . The action  $\beta$  of  $\mathbb{Z}_2$  on  $S^4$  is by reflection

$$(\sigma_2, (x_1, \dots, x_5)) \longrightarrow (x_1, -x_2, x_3, -x_4, x_5).$$

The group  $\mathbb{Z}_2$  also acts on  $\mathbb{R}^2$  by reflection

$$\rho : \sigma_2 \longrightarrow \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

On  $\mathbb{R}^2$ , for  $\theta \in \mathbb{R}$ , consider the same symplectic form  $J = \theta dx_1 \wedge dx_2$ . It is easy to check that the actions  $\alpha, \beta, \rho$  satisfy Eq. (2). Consider the algebra  $C(S^4)$  of continuous functions on  $S^4$ . Rieffel's construction defines a deformation  $C(S^4_\theta)$  of  $C(S^4)$  by  $J$  and the action  $\alpha$ , which is the  $\theta$ -deformation [3] introduced by Connes and Landi. As is explained in Sec. 1,  $\mathbb{Z}_2$  acts strongly continuously on  $C(S^4_\theta)$ . Theorem 1.2 states that

$$K_\bullet(C(S^4) \rtimes \mathbb{Z}_2) = K_\bullet(C(S^4_\theta) \rtimes \mathbb{Z}_2).$$

The  $K$ -theory of  $C(S^4) \rtimes \mathbb{Z}_2$  can be computed [7] topologically by the Grothendieck group of  $\mathbb{Z}_2$ -equivariant vector bundles on  $S^4$ .

Notice that the quotient  $S^4/\mathbb{Z}_2$  is an orbifold homeomorphic to  $S^4$ . As an orbifold,  $S^4/\mathbb{Z}_2$  [10] has a good covering  $\{U_i\}$  such that each  $U_i$  and any none empty finite intersection  $U_{i_1} \cap \dots \cap U_{i_k}$  is a quotient of a finite group action on  $\mathbb{R}^4$ . Such a good covering allows to compute the topological  $\mathbb{Z}_2$ -equivariant  $K$ -theory of  $S^4$  by the Čech cohomology on  $S^4/\mathbb{Z}_2$  of the sheaf  $\mathcal{K}_{\mathbb{Z}_2}^\bullet$  introduced by Segal [11]. The restriction of  $\mathcal{K}_{\mathbb{Z}_2}^\bullet$  to an open chart  $U$  of  $S^4/\mathbb{Z}_2$  is defined to be the  $\mathbb{Z}_2$ -equivariant  $K$ -theory of  $\pi^{-1}(U)$  with  $\pi$  the canonical projection  $S^4 \rightarrow S^4/\mathbb{Z}_2$ . Locally, when  $U$  is sufficiently small, we can compute  $K_{\mathbb{Z}_2}^\bullet(U)$  to be  $K^\bullet(\pi^{-1}(U)^{\sigma_2}) \oplus K^\bullet(U)$ , where  $\pi^{-1}(U)^{\sigma_2}$  is the  $\sigma_2$ -fixed point submanifold. When  $\bullet = 0$ , it is equal to  $\mathbb{Z}|_{\pi^{-1}(U)^{\sigma_2}} \oplus \mathbb{Z}_U$ , and when  $\bullet = 1$ , it is zero. Gluing this local computation by the Mayer-Vietoris sequence, we conclude that

$$K_0(C(S^4_\theta) \rtimes \mathbb{Z}_2) = \mathbb{Z}^4, \quad K_1(C(S^4_\theta) \rtimes \mathbb{Z}_2) = 0.$$

**Remark 3.1.** We observe that in the above example, the group  $\mathbb{Z}_2$  is not essential. Our computations generalize to  $K_\bullet(C^\infty(S^4_\theta) \rtimes \mathbb{Z}_i)$ , for  $i = 3, 4, 6$ .

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